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Boundedness properties for Sobolev inner products

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Abstract

Sobolev orthogonal polynomials with respect to measures supported on subsets of the complex plane are considered. The connection between the following properties is studied: the multiplication operator $\mathcal{M}p(z) = zp(z)$ defined on the space $\mathbb{P}$ of algebraic polynomials with complex coefficients is bounded with respect to the norm defined by the Sobolev inner product, the supports of the measures are compact and the zeros of the orthogonal polynomials lie in a compact subset of the complex plane. In particular, we prove that the boundedness of the multiplication operator $\mathcal{M}$ always implies the compactness of the supports.

1. Introduction

It is well known that inner products defined on the linear space of polynomials $\mathbb{P}$ as

$$\langle p, q \rangle = \int p\bar{q}\, d\mu, \tag{1}$$

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where μ is a positive Borel measure with $\operatorname{supp}\mu\subseteq\mathbb{R}$ are characterized for the symmetry of the multiplication operator

$$\mathscr{M}:\mathbb{P}\to\mathbb{P},$$

$$\mathscr{M}(p)=zp(z); \tag{2}$$

that is, an inner product defined on $\mathbb{P}$ satisfies $\langle zp(z),q(z)\rangle=\langle p(z),zq(z)\rangle$ for all $p,q\in\mathbb{P}$, if and only if $\langle\,,\rangle$ is defined as in (1) where μ is a positive measure with $\operatorname{supp}\mu\subseteq\mathbb{R}$.

The fact that the multiplication operator is symmetric for $\langle\,,\rangle$ has a number of important consequences. The three-term recurrence relation for the sequence of orthonormal polynomials with respect to $\langle\,,\rangle$ is the most important (they are, in fact, equivalent properties, see [2, Introd.]). Another important consequence is also the equivalence of the following three properties:

(i) The multiplication operator $\mathscr{M}p(z)=zp(z)$ is bounded with respect to the norm defined by the inner product.
(ii) The zeros of the orthogonal polynomials with respect to the inner product lie in a compact set.
(iii) The support of the measure μ is compact.

Actually, (i) always implies (ii) (see [4]), and (i) is equivalent to (iii) also for the more general case of inner products as in (1) where μ is now a measure with $\operatorname{supp}\mu\subseteq\mathbb{C}$. But even in that case, some examples can be easily found showing that (ii) does not imply (iii) (see Example 1 in the next section).

The aim of this paper is to explore the connection between the above properties (i), (ii) and (iii) for Sobolev inner products.

Let μ be a finite positive Borel measure supported in a subset of the complex plane. Let $W=(w_{i,j})_{i,j=0}^N$ be a matrix of integrable functions with respect to μ which it is positive semidefinite μ a.e., i.e, $W(z)=(w_{i,j}(z))_{i,j=0}^N$ is positive semidefinite for any complex number z except for at most a set of μ measure zero. We consider the following Sobolev inner product defined on the space $\mathbb{P}$ of algebraic polynomials with complex coefficients

$$\langle p,q\rangle=\sum_{k=0}^{N}\int(p(z),p'(z),\dots,p^{(N)}(z))\,W(z)\begin{pmatrix}\overline{q(z)}\\ \overline{q'(z)}\\ \vdots\\ \overline{q^{(N)}(z)}\end{pmatrix}d\mu(z). \tag{3}$$

As usual, $f^{(k)}$ denotes the kth derivative of a function f and the norm of a polynomial $p\in\mathbb{P}$ with respect to the Sobolev inner product is just $\|p\|=\sqrt{\langle p,p\rangle}$.

We will assume that $\operatorname{supp}\mu=\bigcup_{k=0}^N\operatorname{supp}w_{k,k}$ contains infinitely many points and that $W(t)$ is positive definite for at least an infinite subset of $\operatorname{supp}\mu$. Therefore, a unique sequence of monic orthogonal polynomials, which will be denoted by $(q_n)_{n\geqslant0}$, is associated with (3). For each $n\in\mathbb{N}$, the degree of q_n

is exactly n. The more studied Sobolev inner product appears when the matrix W is taken to be diagonal: $W = \mathrm{diag}\{w_0, \ldots, w_N\}$; then the Sobolev inner product reduces to

$$\langle p, q \rangle = \sum_{k=0}^{N} \int p^{(k)}(z)\overline{q^{(k)}(z)} w_k(z)\, d\mu(z) = \sum_{k=0}^{N} \langle p^{(k)}, q^{(k)} \rangle_{L_2(w_k\, d\mu)}, \tag{4}$$

with the usual Sobolev norm

$$||p|| = \left(\sum_{k=0}^{N} ||p^{(k)}||_{w_k}^2 \right)^{1/2}, \quad \text{where } ||p||_{w_k}^2 = \int |p(z)|^2 w_k\, d\mu(z), \ \ k = 0, \ldots, N. \tag{5}$$

Sobolev orthogonal polynomials have attracted much attention in the past decade (especially the diagonal case). Many papers on the subject deal with the algebraic aspects of the theory. In this direction, we call attention to the recent papers [4,5] which deal with the zero distribution of Sobolev orthogonal polynomials assuming that μ has compact support, and [3] for zero location of nonstandard orthogonal polynomials including that of Sobolev type.

The key concept to establish the boundedness of the multiplication operator for Sobolev inner product is that of the sequentially dominated measures which was introduced by Lopez and Pijeira in [5]:

Definition 1. *Given a measure μ with support in the complex plane and nonnegative functions $w_k \in L^1(\mu)$, $k = 0, \ldots, N$, the measures $\mu_k = w_k\, d\mu$, $k = 0, \ldots, N$, are said to be sequentially dominated if they satisfy $w_k/w_{k-1} \in L^\infty(\mu)$, $k = 1, \ldots, N$.*

The property of sequential domination for measures with compact support always implies the boundedness of the multiplication operator for the Sobolev inner product defined by those measures [4, Section 2]. This result was originally given in [5, Theorem 1] for the case $\mathrm{supp}\, \mu \subseteq \mathbb{R}$; moreover Rodríguez gave in [6] the following characterization for the boundedness of the multiplication operator for Sobolev inner products:

Proposition 1 (Rodríguez [6, Theorem 4.1]). *Let $\langle\, , \rangle$ be the Sobolev inner product defined by* (4) *where μ has compact support. The multiplication operator is bounded for this inner product if and only if the norm* (5) *is equivalent to the norm defined by a family of measures $\tilde{\mu}_k = \tilde{w}_k\, d\mu$, $k = 0, \ldots, N$, which is sequentially dominated. Moreover, in this case we can take the nonnegative functions $\tilde{w}_k \in L^1(\mu)$ to be $\tilde{w}_k = w_k + w_{k+1} + \cdots + w_N$.*

We prove in this note that the boundedness of any power of the multiplication operator for a general Sobolev inner product as in (3) always implies the compactness of the support of the measure (Section 2); we also give some examples showing that the converse is not true, and that $\mathcal{M}^2$ can be bounded and $\mathcal{M}$ not. It still remains as an open question whether the properties (ii) and (iii) above are also equivalent for Sobolev inner products with $\operatorname{supp}\mu \subseteq \mathbb{R}$ and to prove or disapprove that (iii) implies (ii) when $\operatorname{supp}\mu \subseteq \mathbb{C}$.

An important tool in our research will be the matrix approach presented in [3] for the location of the zeros of orthogonal polynomials with respect to nonstandard inner products (see Section 3). Using it, we give the following characterization of sequentially dominated measures:

Proposition 2. *Let μ be a positive measure with support in the complex plane and consider nonnegative functions $w_k \in L^1(\mu)$, $k = 0, \ldots, N$. Then, the measure μ has compact support and the measures $\mu_k = w_k\, d\mu$, $k = 0, \ldots, N$, are sequentially dominated if and only if there exists $\eta > 0$ such that the matrix $\mathscr{A}(t) := \eta W(t) - \Gamma(t) W(t) \Gamma^*(t)$ is positive semidefinite for μ almost every $t \in \operatorname{supp}\mu$, where $W = \operatorname{diag}\{w_0, \ldots, w_N\}$ and*

$$\Gamma(t) = \begin{pmatrix} t & 1 & 0 & \ldots & & 0 \\ 0 & t & 2 & 0 & \ldots & 0 \\ 0 & 0 & \ddots & \ddots & & \vdots \\ \vdots & & & & & \\ & \ldots & & & t & N \\ 0 & \ldots & & & 0 & t \end{pmatrix}.$$

2. Compactness of the measure's support

We will now prove that the boundedness of any power of the multiplication operator for a general Sobolev inner product always implies the compactness of the measure's support.

Theorem 1. *Let $\langle\, , \rangle$ be the Sobolev inner product defined by* (3). *If for some $k \geqslant 1$, $\|\mathscr{M}^k\| < \infty$ then $\bigcup_{k=0}^{N} \operatorname{supp}(w_{k,k}\, d\mu) \subset \{z \in \mathbb{C}\colon |z| \leqslant \|\mathscr{M}^k\|^{1/k}\}$. (Actually, it is enough to assume $\mathscr{M}^k$ bounded on the sequence of monomials $(z^n)_n$).*

Proof. Notice that $\|z^{kn}\| \leqslant \|\mathscr{M}^k\|^n \cdot \|z^0\|$, $n \geqslant 1$, which implies that $\|z^{kn}\|^2 \leqslant \|\mathscr{M}^k\|^{2n} C$.

Write μ_k, $k = 0, \ldots, N$, for the measure $d\mu_k = w_{k,k}\, d\mu$ and suppose that there exists $\varepsilon > 0$ and k_0: $0 \leqslant k_0 \leqslant N$ such that $\mu_{k_0}(\{z\colon |z| > \|\mathscr{M}^k\|^{1/k} + \varepsilon\}) > 0$; we take k_0 the biggest one satisfying that property. Write $A = \{z\colon |z| > \|\mathscr{M}^k\|^{1/k} + \varepsilon\}$ and $\alpha_{k,n,i} =$

$kn(kn-1)\cdots(kn-i+1)$, $i=1,\dots,N$; hence,

$$
\begin{aligned}
\|\mathscr{M}^k\|^{2n}C \geqslant \|z^{kn}\|^2 &= \int (z^{kn},(z^{kn})',\dots,(z^{kn})^{(N)})\,W(z)\begin{pmatrix}\overline{z^{kn}}\\ \overline{(z^{kn})'}\\ \vdots\\ \overline{(z^{kn})^{(N)}}\end{pmatrix} d\mu(z)\\
&= \int |z|^{2(kn-N)}(z^N,\alpha_{k,n,1}z^{N-1},\dots,\alpha_{k,n,N})\,W(z)\begin{pmatrix}\overline{z^N}\\ \overline{\alpha_{k,n,1}z^{N-1}}\\ \vdots\\ \overline{\alpha_{k,n,N}}\end{pmatrix} d\mu(z)\\
&\geqslant \int_A |z|^{2(kn-N)}(z^N,\alpha_{k,n,1}z^{N-1},\dots,\alpha_{k,n,N})\,W(z)\begin{pmatrix}\overline{z^N}\\ \overline{\alpha_{k,n,1}z^{N-1}}\\ \vdots\\ \overline{\alpha_{k,n,N}}\end{pmatrix} d\mu(z)\\
&\geqslant (\|\mathscr{M}^k\|^{1/k}+\varepsilon)^{2(kn-N)}\int_A (z^N,\alpha_{k,n,1}z^{N-1},\dots,\alpha_{k,n,N})\,W(z)\\
&\quad\times\begin{pmatrix}\overline{z^N}\\ \overline{\alpha_{k,n,1}z^{N-1}}\\ \vdots\\ \overline{\alpha_{k,n,N}}\end{pmatrix} d\mu(z);
\end{aligned}
$$

from the choice of k_0 follows that $\mu_i(A)=0$ for $i>k_0$, the last formula gives then

$$
\begin{aligned}
\|\mathscr{M}^k\|^{2n}C &\geqslant (\|\mathscr{M}^k\|^{1/k}+\varepsilon)^{2(kn-N)}\int_A (z^N,\dots,\alpha_{k,n,k_0}z^{N-k_0},0,\dots,0)\,W(z)\\
&\quad\times\begin{pmatrix}\overline{z^N}\\ \vdots\\ \overline{\alpha_{k,n,k_0}z^{N-k_0}}\\ 0\\ \vdots\\ 0\end{pmatrix} d\mu(z)\\
&\geqslant \alpha_{k,n,k_0}^2(\|\mathscr{M}^k\|^{1/k}+\varepsilon)^{2(kn-N)}\int_A \left(\frac{z^N}{\alpha_{k,n,k_0}},\dots,z^{N-k_0},0,\dots,0\right)W(z)
\end{aligned}
$$

$$\times \begin{pmatrix} \overline{z^N/\alpha_{k,n,k_0}} \\ \vdots \\ \overline{z^{N-k_0}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} d\mu(z). \tag{6}$$

Since for $0 \leqslant i < k_0$, the sequence $\alpha_{k,n,i}/\alpha_{k,n,k_0}$ tends to 0 as n tends to ∞, we get that

$$\lim_n \int_A \left(\frac{z^N}{\alpha_{k,n,k_0}}, \ldots, z^{N-k_0}, 0, \ldots, 0\right) W(z) \begin{pmatrix} \overline{z^N/\alpha_{k,n,k_0}} \\ \vdots \\ \overline{z^{N-k_0}} \\ 0 \\ \vdots \\ 0) \end{pmatrix} d\mu(z)$$

$$= \int_A |z|^{2(N-k_0)}\, d\mu_{k_0}$$
$$\geqslant (||\mathscr{M}^k||^{1/k} + \varepsilon)^{2(N-k_0)} \mu_{k_0}(A).$$

Hence, by taking $\delta > 0$ small enough (6) gives

$$C||\mathscr{M}^k||^{2n} \geqslant \frac{\delta\alpha^2_{k,n,k_0}}{(||\mathscr{M}^k||^{1/k} + \varepsilon)^{2k_0}} (||\mathscr{M}^k||^{1/k} + \varepsilon)^{2kn},$$

that is,

$$C \geqslant \frac{\delta\alpha^2_{k,n,k_0}}{(||\mathscr{M}^k||^{1/k} + \varepsilon)^{2k_0}} \left(\frac{(||\mathscr{M}^k||^{1/k} + \varepsilon)^{2k}}{||\mathscr{M}^k||^2}\right)^n;$$

which is a contradiction because the expression on the right of this inequality tends to infinity when n tends to infinity. □

As the following example shows, if the support of the measure is not contained in the real line, the boundedness of the set of zeros of the orthogonal polynomials does not imply the boundedness of any power of the multiplication operator, even in the standard case:

Example 1. Consider the Sobolev inner product (4) for $N = 0$, the measure $\mu_0 = \sum_{k \geqslant 1} \frac{\theta_k}{2^k}$, with $\theta_k = \frac{d\theta}{2\pi k}$, where θ is the Lebesgue measure supported in the circle of center 0 and radio k, which reduces to the standard product

$$\langle p, q \rangle = \frac{1}{2\pi} \int_{\mathbb{C}} p(z)^n \overline{p(z)}^m \, d\mu_0.$$

In this case, the corresponding orthonormal polynomials are the monomials z^n, $n \geqslant 0$. We have that $\mu_0(\mathbb{C}) = \sum_{k=1}^{\infty} \frac{\theta_k(\mathbb{C})}{2^k} = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$. The support of μ_0 is unbounded, which according to Theorem 1 implies that any power of the multiplication operator is unbounded despite the boundedness of the corresponding set of zeros.

In order to complete Theorem 1, some examples will show that in contrast with the case of standard orthogonality:

- if μ is supported in the real line, neither the compactness of the support of the measure μ in (4) nor the boundedness of the set of zeros of the corresponding Sobolev orthogonal polynomials imply the boundedness of any power of the multiplication operator;
- $\mathscr{M}$ can be unbounded and $\mathscr{M}^2$ bounded.

Example 2. Consider the Sobolev inner product (4) for $N = 1$, the measure $\mu = \lambda + \delta_{1/2}$, where λ is the Lebesgue measure $d\lambda = dx$ with support $[\frac{1}{2}, 1]$ and the functions $w_0(x) = \chi_{(\frac{1}{2},1]}(x)$ and $w_1(x) = \chi_{\{1/2\}}(x)$, which reduces to the product

$$\langle p, q \rangle = \int_{[\frac{1}{2},1]} p(x)\bar{q}(x)\,dx + p'(1/2)\overline{q'}(1/2). \tag{7}$$

Proposition 3.3 of [1] shows that the set of zeros of the orthogonal polynomials with respect to this inner product are bounded.

We now prove that any power of the multiplication operator is unbounded. Take the sequence of polynomials $t_n(x) = ((x-1/2)^2 - 1)^n$, $n \geqslant 0$; they satisfy the inequalities, $n \geqslant 0$,

$$\langle x^k t_n, x^k t_n \rangle = \int_{1/2}^{1} x^{2k}((x-1/2)^2 - 1)^{2n}\,dx + k^2\left(\frac{1}{2}\right)^{2k-2} \geqslant k^2\left(\frac{1}{2}\right)^{2k-2},$$

and

$$\langle t_n, t_n \rangle = \int_{1/2}^{1} ((x-1/2)^2 - 1)^{2n}\,dx.$$

But $|(x-1/2)^2 - 1| < 1$, $x \in (1/2, 1]$, hence $((x-1/2)^2 - 1)^{2n}$ tends to 0 on $(1/2, 1]$ as n tends to ∞, and so $\langle t_n, t_n \rangle$ tends to 0 as n tends to ∞; this implies that $\mathscr{M}^k$ is not bounded.

Example 3. We now show another example of Sobolev inner product for which $\mathscr{M}$ is not bounded, $\mathscr{M}^2$ is bounded and whose orthogonal polynomials have bounded zeros. In this case, the measure μ_1 does not reduce to a Dirac delta as in the previous example: moreover the set $\mathrm{supp}(\mu_1) \backslash \mathrm{supp}(\mu_0)$ is infinite. Indeed, we take $\mu = \lambda + \delta_0 + \delta_{1/4} + \delta_{-1/4}$, where λ is now the Lebesgue measure $d\lambda = dx$ with support

$$[-1, -\tfrac{1}{4}] \cup [\tfrac{1}{4}, 1]$$

and the functions $w_0(x) = \chi_{[-1,-\frac{1}{2}]\cup[\frac{1}{2},1]\cup\{-1/4,1/4\}}(x)$ and $w_1(x) = \chi_{[-1,-1/4)\cup\{0\}\cup(1/4,1]}(x)$, which reduces to the product

$$\langle p, q\rangle_1 = \int_{[-1,-\frac{1}{2}]\cup[\frac{1}{2},1]} p(x)\bar{q}(x)dx + p(-1/4)\overline{q(-1/4)} + p(1/4)\overline{q(1/4)} + p'(0)\overline{q'}(0) + \int_{[-1,-\frac{1}{4}]\cup[\frac{1}{4},1]} p'(x)\overline{q'(x)}\,dx; \tag{8}$$

we then have that $\mathrm{supp}(\mu_1)\backslash\mathrm{supp}(\mu_0) = (-1/2,-1/4)\cup\{0\}\cup(1/4,1/2)$. We first prove that the operator of multiplication by t is not bounded for this inner product, but the operator of multiplication by t^2 is however bounded. To do this consider the inner product $\langle , \rangle_2$ defined as $\langle , \rangle_1$ but removing the point 0 from the support of w_1, that is

$$\langle p, q\rangle_2 = \int_{[-1,-\frac{1}{2}]\cup[\frac{1}{2},1]} p(x)\bar{q}(x)dx + p(-1/4)\overline{q(-1/4)} + p(1/4)\overline{q(1/4)} + \int_{[-1,-\frac{1}{4}]\cup[\frac{1}{4},1]} p'(x)\overline{q'(x)}\,dx. \tag{9}$$

By writing $p(t) = p(1/4) + \int_{1/4}^{t} p'(x)\,dx$, $1/4\leqslant t$, we deduce for $1/4\leqslant t\leqslant 1$ that

$$|p(t)|\leqslant|p(1/4)| + \int_{1/4}^{1} |p'(x)|\,dx \leqslant |p(1/4)| + \frac{\sqrt{3}}{2}||p'||_{L^2([1/4,1])};$$

analogously, for $-1\leqslant t\leqslant -1/4$

$$|p(t)|\leqslant|p(-1/4)| + \frac{\sqrt{3}}{2}||p'||_{L^2([-1,-1/4])}.$$

This gives that

$$\begin{aligned}||p + xp'||_{L^2([-1,-1/4]\cup[1/4,1])} &\leqslant ||p||_{L^2([-1,-1/4]\cup[1/4,1])} + ||xp'||_{L^2([-1,-1/4]\cup[1/4,1])}\\ &\leqslant \frac{\sqrt{3}}{2}(||p||_{L^\infty([-1,-1/4])} \\ &\quad + ||p||_{L^\infty([1/4,1])}) + ||p'||_{L^2([-1,-1/4]\cup[1/4,1])}\\ &\leqslant \frac{\sqrt{3}}{2}(|p(-1/4)| + |p(1/4)|) + \frac{7}{4}||p'||_{L^2([-1,-1/4]\cup[1/4,1])}.\end{aligned}$$

From where it follows that the operator of multiplication by t is bounded for $\langle\, ,\, \rangle_2$:

$$\begin{aligned}\langle xp, xp\rangle_2 &= \|xp\|^2_{L^2([-1,-1/2]\cup[1/2,1])} + \frac{1}{16}(|p(-1/4)|^2 + |p(1/4)|^2)\\ &\quad + \|p + xp'\|^2_{L^2([-1,-1/4]\cup[1/4,1])}\\ &\leqslant \|p\|^2_{L^2([-1,-1/2]\cup[1/2,1])} + \frac{1}{16}(|p(-1/4)|^2 + |p(1/4)|^2)\\ &\quad + \left(\frac{\sqrt{3}}{2}|p(-1/4)| + \frac{\sqrt{3}}{2}|p(1/4)| + \frac{7}{4}\|p'\|_{L^2([-1,-1/4]\cup[1/4,1])}\right)^2\\ &\leqslant \|p\|^2_{L^2([-1,-1/2]\cup[1/2,1])} + \frac{1}{16}(|p(-1/4)|^2 + |p(1/4)|^2)\\ &\quad + 3\left(\frac{3}{4}|p(-1/4)|^2 + \frac{3}{4}|p(1/4)|^2 + \frac{49}{16}\|p'\|^2_{L^2([-1,-1/4]\cup[1/4,1])}\right)\\ &\leqslant \frac{147}{16}(\|p\|^2_{L^2([-1,-1/2]\cup[1/2,1])} + |p(-1/4)|^2 + |p(1/4)|^2\\ &\quad + \|p'\|^2_{L^2([-1,-1/4]\cup[1/4,1])})\\ &= \frac{147}{16}\langle p, p\rangle_2.\end{aligned}$$

As a consequence, we have that the operator of multiplication by t^2 is bounded for $\langle\, ,\, \rangle_1$:

$$\langle x^2p, x^2p\rangle_1 = \langle x^2p, x^2p\rangle_2 \leqslant \left(\tfrac{147}{16}\right)^2\langle p, p\rangle_2 \leqslant \left(\tfrac{147}{16}\right)^2\langle p, p\rangle_1.$$

However, the operator of multiplication by t is not bounded for $\langle\, ,\, \rangle_1$: indeed, taking $t_n(x) = (x^2-1)^n$, an easy calculation gives

$$\langle xt_n, xt_n\rangle_1 \geqslant 1, \quad n \geqslant 0,$$

and

$$\langle t_n, t_n\rangle \leqslant \left(\tfrac{3}{4}\right)^{2n} + 2\left(\tfrac{15}{16}\right)^{2n} + 6n^2\left(\tfrac{15}{16}\right)^{2n-2}, \quad n \geqslant 2.$$

This shows that the operator of multiplication by t is not bounded for $\langle\, ,\, \rangle_1$.

We now prove that, however, the set of zeros of the orthogonal polynomials p_n, $n \geqslant 0$, with respect to the Sobolev inner product $\langle\, ,\, \rangle_1$, is bounded. Indeed, the symmetry of this Sobolev inner product with respect to the origin shows that p_{2n} is an even polynomial and p_{2n+1} is odd. As a consequence if a is a zero of p_n then so is $-a$. If a is a zero of p_n, we can then write $p_n(z) = (z^2 - a^2)q$, from where it follows that

$$0 = \langle p_n, q\rangle_1 = \langle z^2q, q\rangle_1 - \langle a^2q, q\rangle_1.$$

Since $\mathscr{M}^2$ is bounded for this inner product, we have that

$$|a|^2\langle q, q\rangle_1 = |\langle z^2q, q\rangle_1| \leqslant \sqrt{\langle z^2q, z^2q\rangle_1\langle q, q\rangle_1} \leqslant \sqrt{\|\mathscr{M}^2\|}\langle q, q\rangle_1,$$

from where we get that the zeros are bounded.

3. Sequentially dominated measures: a matrix approach

We prove in this section the characterization of sequentially dominated measures given in Proposition 2 (see the Introduction). To do this, we need to introduce the matrix approach presented in [3] for the location of zeros of Sobolev orthogonal polynomials.

Let us consider an inner product of the form

$$\langle p, q\rangle_W = \int (T_0(p), \dots, T_N(p)) W(z) (T_0(q), \dots, T_N(q))^* \, d\mu(z), \tag{10}$$

where (i) $T_0, \dots, T_N$ are linear operators in the space $\mathbb{P}$; (ii) $W(z)$ is a positive definite matrix of integrable functions with respect to the positive measure μ supported on a subset of the complex plane.

Here, we will assume that W is a diagonal matrix

$$W = \operatorname{diag}\{w_0, w_1, \dots, w_N\}, \qquad w_k \in L^1(\mu), \;\; w_k \geqslant 0, \quad k = 0, 1, \dots, N,$$

and $T_0 = I,\; T_1 = D,\; T_2 = D^2, \dots, T_N = D^N$, where D^j represents the differential operator of order j. Therefore, the inner product (10) reduces to the Sobolev inner product (4).

The main advantage of this matrix approach is that it allows to represent the multiplication operator in terms of a simple matrix product. Indeed,

$$(zp, (zp)' \dots, (zp)^{(N)}) = (p, p', \dots, p^{(N)})\Gamma,$$

where

$$\Gamma = \begin{pmatrix} z & 1 & 0 & \dots & & 0 \\ 0 & z & 2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & & \vdots \\ \vdots & & & & & \\ & \dots & & & z & N \\ 0 & \dots & & & 0 & z \end{pmatrix}.$$

Since

$$\langle zp, zp\rangle_W = \int (p, p', \dots, p^{(N)})[\Gamma W \Gamma^*](p, p', \dots, p^{(N)})^* \, d\mu,$$

we find that the multiplication operator is bounded if and only if there exists $\eta > 0$ such that

$$\int (p, p', \dots, p^{(N)})[\eta W - \Gamma W \Gamma^*](p, p', \dots, p^{(N)})^* d\mu \geqslant 0, \quad \text{for all } p \in \mathbb{P}. \tag{11}$$

Thus, the following sufficient condition for the boundedness of the multiplication operator holds: there exists $\eta > 0$ such that

$$\eta W - \Gamma W \Gamma^* \geqslant 0, \quad \mu \text{ a.e.} \tag{12}$$

However, this is not a necessary condition for the boundedness of the multiplication operator. Indeed, the inner product $\langle\, ,\, \rangle_2$ of Example 3 is a counterexample; we have proved that the operator of multiplication by t is bounded for this inner product. But

$$A(t) = \eta W - \Gamma W \Gamma^* = \begin{pmatrix} (\eta - t^2)w_0(t) - w_1(t) & -tw_1(t) \\ -tw_1(t) & (\eta - t^2)w_1(t) \end{pmatrix};$$

if we take $t \in (-1/2, -1/4) \cup (1/4, 1/2)$, the matrix $A(t)$ reduces to

$$A(t) = \begin{pmatrix} -1 & -t \\ -t & (\eta - t^2) \end{pmatrix}$$

which for any $\eta > 0$ is not positive semidefinite.

However, the property (12) turns out to be equivalent to the sequential domination of the measures $\mu_k = w_k\, d\mu$, $k = 0, \dots, N$, as we prove now.

Proof of Proposition 2. We start by proving that if there exists $\eta > 0$ such that the matrix $\mathscr{A} = \mathscr{A}(\eta) = \eta W - \Gamma W \Gamma^*$ is positive semidefinite μ a.e., then the measures $\mu_k = w_k\, d\mu$, $k = 0, \dots, N$, which define the Sobolev inner product (4) are sequentially dominated and $\operatorname{supp} \mu = \bigcup_{k=0}^N \operatorname{supp} w_k$ is a compact set of the complex plane.

It is easy to find the following expression for the matrix $\mathscr{A} = \eta W - \Gamma W \Gamma^*$:

$$\mathscr{A} = \begin{pmatrix} \alpha_0(z) & -\overline{\beta_1}(z) & 0 & \dots & & 0 \\ -\beta_1(z) & \alpha_1(z) & -\overline{\beta_2}(z) & 0 & & \vdots \\ 0 & -\beta_2(z) & \alpha_2(z) & -\overline{\beta_3}(z) & 0 & \dots \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & & & 0 \\ & & & & \alpha_{N-1} & -\overline{\beta_N}(z) \\ 0 & \dots & & 0 & -\beta_N(z) & \alpha_N(z) \end{pmatrix},$$

where

$$\alpha_j(z) = \begin{cases} (\eta - |z|^2)w_j - (j+1)^2 w_{j+1}, & j = 0, 1, \dots, N-1, \\ (\eta - |z|^2)w_N, & j = N, \end{cases}$$

$$\beta_j(z) = jzw_j(z), \quad j = 1, \dots, N.$$

Let $\eta > 0$ be such that the matrix $\mathscr{A}$ is positive semidefinite μ a.e. Then, for all $j = 0, 1, \dots, N$, in particular it holds $\alpha_j(z) \geqslant 0$ μ a.e. For $j = N$ this gives $(\eta - |z|^2)w_N(z) \geqslant 0$, μ a.e.; thus, we can assume that the function $w_N(z)$ is supported in $\{z : |z| \leqslant \sqrt{\eta}\}$. For $j = 0, \dots, N-1$, we also have

$$(\eta - |z|^2)w_j(z) - (j+1)^2 w_{j+1}(z) \geqslant 0, \quad \mu \text{ a.e.} \tag{13}$$

Thus, we also deduce that the functions $w_j(z)$, $j = 0, \ldots, N-1$, can be supported in $\{z\colon |z| \leqslant \sqrt{\eta}\}$. Taking now into account that $\operatorname{supp} \mu = \bigcup_{k=0}^{N} \operatorname{supp} w_k$, we find that the support of the measure μ is compact.

From (13) we also conclude that μ a.e. if $w_j(z) = 0$ then $w_{j+1}(z) = 0$, thus on the support of μ

$$\frac{w_{j+1}(z)}{w_j(z)} \leqslant \frac{(\eta - |z|^2)}{(j+1)^2} \leqslant \tilde{\eta}, \quad j = 0, \ldots, N-1;$$

which implies that the measures $\mu_k = w_k\, d\mu$, $k = 0, \ldots, N$, are sequentially dominated.

In order to prove the converse result, we assume that the measures $\mu_k = w_k\, d\mu$, $k = 0, \ldots, N$, which define the Sobolev inner product (4) are sequentially dominated and $\operatorname{supp} \mu = \bigcup_{k=0}^{N} \operatorname{supp} w_k$ is a compact set of the complex plane. Then we have that there exists a constant $c > 0$ such that for $j = 0, \ldots, N-1$,

$$w_{j+1}(z) \leqslant c w_j(z) \mu \text{ a.e.} \tag{14}$$

Notice that

$$\Gamma(z) W(z) \Gamma(z)^* + \Gamma(-z) W(z) \Gamma(-z)^* = 2 \operatorname{diag}(\Gamma(z) W(z) \Gamma(z)^*)$$

and hence $\mathscr{A}(\eta)$ will be positive semidefinite μ a.e. for some η if this is true for $\tilde{\mathscr{A}} = \tilde{\mathscr{A}}(\eta) = \eta W(z) - 2 \operatorname{diag}(\Gamma(z) W(z) \Gamma(z)^*)$.

For the matrix $\tilde{\mathscr{A}}$, we have the expression

$$\begin{aligned}\tilde{\mathscr{A}} &= \operatorname{diag}\{w_0(\eta - 2|z|^2) - 2w_1, \ldots, w_j(\eta - 2|z|^2) \\ &\quad - 2(j+1)^2 w_{j+1}, \ldots, w_N(\eta - 2|z|^2)\}, \quad j = 0, 1, \ldots, N-1.\end{aligned}$$

Let M be such that $\operatorname{supp} \mu \subset \{z\colon |z| \leqslant M\}$. We can choose η large enough (independently of z) such that $\eta - 2(N^2 c + M^2) \geqslant 0$ and hence, taking (14) into account, the matrix $\tilde{\mathscr{A}}$ will be positive semidefinite μ a.e. $\square$

As a corollary, it follows from Proposition 2 and (11) that sequential domination of measures with compact support implies the boundedness of the multiplication operator, which gives an alternative proof to the result of López–Pijeira in [5, Theorem 1]:

Corollary 1. *Let μ be a positive measure with compact support in the complex plane and consider the Sobolev inner product $\langle\,,\,\rangle$ defined by* (4). *If the measures $\mu_k = w_k\, d\mu$, $k = 0, \ldots, N$, are sequentially dominated then there exists $\eta > 0$ such that $\langle zp, zp \rangle \leqslant \eta \langle p, p \rangle$, $p \in \mathbb{P}$.*

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Appendix

The purpose of this appendix is, for the sake of completeness, to present a simplified proof of the characterization of the boundedness of the multiplication operator for Sobolev inner products given by Rodríguez in [6, Theorem 4.1] (see Proposition 1 in the Introduction of this paper). The proof uses the same arguments given in [6], which also work for measures having support on the complex plane.

We are going to show that the multiplication operator is bounded if and only if the norm (5) is equivalent to the norm defined by the matrix

$$\widetilde{W} = \operatorname{diag}\{w_0 + w_1 + \cdots + w_N, w_1 + w_2 + \cdots + w_N, \ldots, w_{N-1} + w_N, w_N\},$$

that is, there exists $\eta > 0$ satisfying

$$||zp||_W \leqslant \eta ||p||_W \quad \text{for all } p \in \mathbb{P} \tag{A.1}$$

if and only if there exists a positive constant C such that

$$||p||_W \leqslant ||p||_{\widetilde{W}} \leqslant C||p||_W. \tag{A.2}$$

The first of the previous inequalities is straightforward taking into account the definition of the matrix $\widetilde{W}$. However, in order to prove the second inequality in (A.2) we have to show that there exists a constant $C > 0$ such that the following inequality takes place:

$$C||p||_W^2 \geqslant ||p||^2_{w_1+w_2+\cdots+w_N} + ||p'||^2_{w_2+w_3+\cdots+w_N} + \cdots + ||p^{(N-1)}||^2_{w_N}. \tag{A.3}$$

Taking into account Theorem 1, we can omit the hypothesis supp μ compact and state the following result:

Theorem A.1. *If the multiplication operator $\mathscr{M}(p(z)) = zp(z)$ is bounded with respect to the Sobolev norm* (5) *then the inequality* (A.2) *holds.*

We point out that the reciprocal assertion in Theorem A.1 is also true assuming supp μ compact: since $\widetilde{W}$ is sequentially dominated, according to the Corollary 1 the multiplication operator is bounded.

Proof. In order to prove Theorem A.1, it remains to show that the inequality (A.3) holds. For this purpose we will make use of the following.

Lemma A.1. *If $||\mathscr{M}|| < \infty$ then for all $p \in \mathbb{P}$ it holds*

$$||p^{(l)}||_{w_j} \leqslant (2||\mathscr{M}||)^{j-l} ||p||_W, \quad 0 \leqslant l \leqslant j \leqslant N. \tag{A.4}$$

Proof. Taking into account the definition of the norm in (5) for $l = j$ one obtains the trivial inequality $||p^{(j)}||_{w_j} \leqslant ||p||_W$. Thus, we can assume $j > l$.

For each $0 \leqslant l < j \leqslant N$ the following inequality takes place:

$$||(zp)^{l+1}||_{w_j} = ||zp^{(l+1)} + (l+1)p^{(l)}||_{w_j} \geqslant (l+1)||p^{(l)}||_{w_j} - ||zp^{(l+1)}||_{w_j},$$

hence,

$$(l+1)||p^{(l)}||_{w_j} \leqslant ||zp^{(l+1)}||_{w_j} + ||(zp)^{(l+1)}||_{w_j}. \tag{A.5}$$

We proceed by induction. At first, let $j - l = 1$. Considering (5) one has for each $j = 1, \ldots, N$ the estimations

$$||(zp)^{(j)}||_{w_j} \leqslant ||zp||_W \leqslant ||\mathcal{M}|| \cdot ||p||_W \quad \forall p \in \mathbb{P}.$$

On the other hand, taking Theorem 1 into account it holds

$$||zp^{(j)}||_{w_j} \leqslant ||\mathcal{M}|| \cdot ||p^{(j)}||_{w_j} \leqslant ||\mathcal{M}|| \cdot ||p||_W.$$

Substituting the previous inequalities in (A.5) one obtains

$$||p^{(j-1)}||_{w_j} \leqslant \frac{2}{j}||\mathcal{M}|| \cdot ||p||_W,$$

thus the lemma holds true for this particular case.

We now assume (A.3) holds for $j - l = k$. We have to prove that

$$||p^{(j-k-1)}||_{w_j} \leqslant (2||\mathcal{M}||)^{k+1}||p||_W. \tag{A.6}$$

Indeed, substituting $l = j - k - 1$ in (A.5) one obtains the inequality

$$(j-k)||p^{(j-k-1)}||_{w_j} \leqslant ||\mathcal{M}|| \cdot ||p^{(j-k)}||_{w_j} + ||(zp)^{(j-k)}||_{w_j}.$$

The inequality (A.6) follows using the induction hypothesis. $\square$

To complete the proof of (A.3), considering (5) and Lemma A.1, for each $l = 0, 1, \ldots, N-1$ we have the estimation

$$||p^{(l)}||^2_{w_{l+1}+w_{l+2}+\cdots+w_N} = \sum_{j=l+1}^{N} ||p^{(l)}||^2_{w_j} \leqslant \sum_{j=l+1}^{N} (2||\mathcal{M}||)^{2(j-l)}||p||^2_W,$$

hence, it holds

$$\sum_{l=0}^{N-1} ||p^{(l)}||^2_{w_{l+1}+w_{l+2}+\cdots+w_N} \leqslant \sum_{l=0}^{N-1} \sum_{j=l+1}^{N} (2||\mathcal{M}||)^{2(j-l)}||p||^2_W. \qquad \square$$

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